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Abstract

We study exact solutions of the position-dependent effective mass Schrödinger equation by considering the new solvable nonlinear oscillator that relates to the isotonic oscillator through the method of point canonical transformations. We construct exactly solvable potentials by choosing physically important position-dependent mass distributions. We also provide the energy spectrum of the bound states and the wavefunctions of the solvable potentials.

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1. Introduction

In a recent paper, Cariñena *et al* [1] considered a non-polynomial one-dimensional quantum potential representing an oscillator, which can be considered as placed in between the harmonic oscillator and the isotonic oscillator, and showed that it is an exactly solvable potential. The aim of this paper is to obtain new solvable potentials that share the eigenspectrum with that of the nonlinear oscillator related with the isotonic oscillator by connecting the latter into the position-dependent effective mass Schrödinger equation (PDMS).

The motivation to do this analysis arises from two reasons. The primary reason comes from the developments in the study of the PDMS and the secondary one comes from the study of inverse square type potentials in physics. The contemporary studies have shown that in a wide variety of physical problems an effective mass depending on the position is of utmost relevance. To name a few problems we cite (i) effective interactions in nuclear physics [2], (ii) carriers and impurities in crystals [3], (iii) quantum dots [4], (iv) quantum liquids [5], (v) semiconductor heterostructures [6] and (vi) in neutron stars [7]. Apart from the experimental analysis, various theoretical aspects of the PDMS have also been investigated in detail recently; for example, exact solvability [8], shape invariance [9], quasi-exact solvability [10], supersymmetric or

intertwining formulation [11], Lie-algebraic approach [12] and Green's approach [13] have been studied widely.

The inverse square-type potentials have been broadly studied in physics under different considerations. For example, it has been shown that they appear in the study of the Bose–Einstein condensates and cosmological models [14–16], and in the solution of time-dependent harmonic or anharmonic oscillators [17–19]. Secondly it has been of interest for physicists for several decades to solve the Schrödinger equation with non-polynomial potentials [20] due to the interesting properties they admit. For example, the isotonic oscillator potential ($V = \frac{\omega^2}{2}x^2 + \frac{g}{2x^2}$; ω and g are constants) at the classical level admits periodic oscillations with the same period [21] (isochronous potential), and at the quantum level it is Schrödinger solvable and the energy spectrum is equidistant [22]. All these facts motivate us to take up this problem.

To obtain the new solvable potentials that share the same eigenspectrum with that of the nonlinear oscillator we relate the underlying Schrödinger equation to the Schrödinger equation with position-dependent mass. We identify new solvable potentials by using the fact that the PDMS and Schrödinger equation with a constant mass share the same energy spectrum and can be transformed into each other through point canonical transformations. In this procedure, one has to consider an exactly solvable potential as a reference potential and construct the exactly solvable target potential by choosing a suitable mass distribution. For our studies, we consider three different mass distributions that are often used in semiconductor physics. The eigenvalues and eigenfunctions of the target potential can also be obtained in this procedure.

The plan of this paper is as follows. In the following section, we briefly recall the eigenvalues and eigenspectrum of the newly solvable nonlinear oscillator that is related to the isotonic oscillator. In section 3, we discuss the method of solving the PDMS. By utilizing the method, we construct new solvable potentials that share the same energy spectrum with that of the nonlinear oscillator in section 4. Finally, we present our conclusions in section 5.

2. A new solvable potential

In [1] Cariñena *et al* have considered the potential

$$V(y) = \frac{1}{2}\left(y^2 + \frac{8(2y^2 - 1)}{(2y^2 + 1)^2}\right) \tag{1}$$

and proved that it is one of the exactly solvable potentials in quantum mechanics. They have also given explicit forms of the energy spectrum and the wavefunctions for this oscillator. Consequently, Fellows and Smith have shown that the potential considered by Cariñena *et al* is a supersymmetric partner potential of the harmonic oscillator [23].

The potential given above is very closely related with the isotonic oscillator:

$$V_{\text{iso}}(y) = \frac{\omega^2}{2}y^2 + \frac{g}{2y^2}, \qquad g > 0. \tag{2}$$

The classical Euler–Lagrange's equation of (2) is nothing but the Ermakov–Pinney equation [24]. The classical and quantum dynamics of equation (2) have been discussed at length in the literature (see, for example, [25] and references therein). Cariñena *et al* have considered a generalized version of the potential related to the isotonic oscillator of the form

$$V_{0a}(y) = \frac{1}{2}\left(\omega^2 y^2 + \frac{2g_a(y^2 - a^2)}{(y^2 + a^2)^2}\right), \qquad g_a > 0 \tag{3}$$

with g_a being a real positive parameter. The authors have argued that the new additional terms can be written as the sum of two centripetal barriers in the complex plane

$$V_{0a}(y) = \frac{1}{2}\left(\omega^2 y^2 + g_a\left(\frac{1}{(y+\mathrm{i}a)^2} + \frac{1}{(y-\mathrm{i}a)^2}\right)\right), \tag{4}$$

so it is a rational potential with two imaginary poles symmetric with respect to the origin. For example, if g_a remains constant then when a goes to zero and to ∞ the potential $V_{0a}(y)$ converges to the isotonic and the harmonic oscillator, respectively.

The authors have considered a particular case, namely, $a^2 = \frac{1}{2}$ and $g_a = 2$ in (3) and solved the associated Schrödinger equation, that is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2} + \frac{m_0}{\hbar^2}\left(2E - \omega^2 y^2 - \frac{8(2y^2-1)}{(2y^2+1)^2}\right)\psi = 0, \tag{5}$$

and obtained the eigenfunctions in terms of $\mathcal{P}$-Hermite functions, namely

$$\psi_n(y) = N_n \frac{\mathcal{P}_n(y)}{(1+2y^2)}\mathrm{e}^{-\frac{y^2}{2}}, \qquad n = 0, 3, 4, \ldots, \tag{6}$$

where the normalization constant is given by

$$N_n = \left(\frac{(n-1)(n-2)}{2^n n!\sqrt{\pi}}\right)^{\frac{1}{2}}, \qquad n = 0, 3, 4, \ldots \tag{7}$$

and the $\mathcal{P}$-Hermite polynomials are defined by

$$\mathcal{P}_n(y) = H_n + 4n H_{n-2} + 4n(n-3)H_{n-4}, \qquad n = 3, 4, 5\ldots, \tag{8}$$

where H_n's are nothing but Hermite polynomials. While solving the Schrödinger equation (5) the authors have taken $\hbar, m_0, \omega = 1$ for ease of notation. We also follow the same in order to make our results compatible with them.

The energy spectrum is given by

$$E_n = -\frac{3}{2} + n. \tag{9}$$

The result shows that the ground state ψ_0 has an energy E_0 which is lower than that of the pure harmonic case [1].

3. Method of solving PDMS

In this section, we recall briefly the point canonical transformation approach to solve the Schrödinger equation with position-dependent mass [11, 26, 27]. For a one-dimensional potential $V(y)$, the time-independent Schrödinger equation with a constant mass is given by

$$-\frac{1}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2} + V(y)\psi(y) = E\psi(y), \tag{10}$$

where we have taken $\hbar = 1$ and the mass $m_0 = 1$. Let us introduce a transformation $y \to x$ through a mapping function $y = f(x)$ and

$$\psi(y) = g(x)\tilde{\psi}(x) \tag{11}$$

in the Schrödinger equation so that the latter becomes

$$\left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\frac{g'}{g} - \frac{f''}{2f'}\right)\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}\left(\frac{g''}{g} - \frac{f''}{f'}\frac{g'}{g}\right) + (f')^2 V(f(x))\right)\tilde{\psi}(x) = (f')^2 E\tilde{\psi}(x), \tag{12}$$

where prime denotes differentiation with respect to x. In the case of the symmetric ordering of the momentum and mass, the one-dimensional Schrödinger equation with position-dependent mass is given by

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{m(x)}\frac{\mathrm{d}\tilde{\psi}(x)}{\mathrm{d}x}\right) + \tilde{V}(x)\tilde{\psi}(x) = \tilde{E}\tilde{\psi}(x). \tag{13}$$

In our analysis, we consider the function $m(x) > 0$. This restriction keeps the functions $\tilde{\psi}(x)$ square integrable [28].

Equation (13) can be rewritten explicitly as

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m'}{2m}\frac{\mathrm{d}}{\mathrm{d}x} + m\tilde{V}(x)\right)\tilde{\psi}(x) = m\tilde{E}\tilde{\psi}(x). \tag{14}$$

Now comparing equations (12) and (14) one observes that equation (12) is identical to equation (14) if the following conditions hold, namely,

$$g(x) = \sqrt{\frac{f'}{m}}, \tag{15}$$

$$\tilde{V}(x) - \tilde{E} = \frac{f'^2}{m}(V(f(x)) - E) - \frac{1}{2m}\left(\frac{g''}{g} - \frac{f''}{f'}\frac{g'}{g}\right). \tag{16}$$

Let us choose $f'^2 = m$ so that equations (15) and (16) now become

$$g(x) = m^{-\frac{1}{4}}(x), \tag{17}$$

$$\tilde{V}(x) = V(f(x)) + \frac{1}{8m}\left(\frac{m''}{m} - \frac{7m'^2}{4m^2}\right). \tag{18}$$

Let us denote the old potential $V(y)$ as the reference potential and the new potential $\tilde{V}(x)$ as the target potential. Since we know the bound state energy spectrum and the wavefunction of the reference potential, E_n and $\psi_n(y)$, respectively, we can now construct the energy spectrum and wavefunctions, $\tilde{E}_n$ and $\tilde{\psi}_n(y)$, respectively, of the target potential $\tilde{V}(x)$ from equations (16) and (18), respectively, that is

$$\begin{aligned}\tilde{E}_n &= E_n, \\ \tilde{\psi}_n(x) &= \frac{1}{g(x)}\tilde{\psi}(x) = m^{\frac{1}{4}}(x)\psi_n(f(x)).\end{aligned} \tag{19}$$

The normalization of $\tilde{\psi}_n(x)$ means that [29, 30]

$$\begin{aligned}\int_{-\infty}^{\infty}|\tilde{\psi}_n(x)|^2\,\mathrm{d}x = 1 \quad &= N_n^2\int_{-\infty}^{\infty}m^{\frac{1}{2}}(x)\psi_n(f(x))^2\,\mathrm{d}x, \\ &= N_n^2\int_{f_{\min}}^{f_{\max}}\psi_n(f(x))^2\,\mathrm{d}f,\end{aligned} \tag{20}$$

where $f_{\min} = f(-\infty)$ and $f_{\max} = f(\infty)$. One may note that the normalization constant in (20) need not be the same as that of the constant mass case. Suppose the lower and upper limits are $-\infty$ and ∞ respectively as in the constant mass case, then the normalization constant assumes the same value which can be seen from our first example in the following section. Suppose the lower and upper limits turned out to be 0 and ∞, then the value of the normalization constant is increased by $\sqrt{2}$ times. We consider this situation in our second example. One may also face the situation in which the lower and upper limits of the above

integral do not fall into either of the above categories. In this case, even though the resultant wavefunctions are square integrable they are not normalizable [29]. For more details about the normalization constant in the position-dependent mass oscillators, one may refer to [29–31].

For a specific mass distribution $m(x)$, one can get eigenvalues and eigenfunctions without solving the Schrödinger equation by using the relations (11), (17), (18) and (19). In the following section, we illustrate the procedure with suitable examples.

4. Applications

In this section, we consider three kinds of position-dependent mass distributions that are often used in semiconductor physics, choose the nonlinear oscillator potential as the reference potential and construct the target potential.

4.1. Case 1

In the nanofabrication of semiconductor devices, one observes quantum wells with very thin layers [32]. The effective mass of an electron (hole) in the thin-layered quantum wells varies with the composition rate. In such systems, the mass of the electron may change with the composition rate which depends on the position. As a consequence, attempts have been made to analyze such PDMS and their underlying properties for a number of potentials and masses. One such mass profile which is found to be useful for studying transport properties in semiconductors is given by [6, 11, 32]

$$m(x) = \frac{(\gamma + x^2)^2}{(1 + x^2)^2},\tag{21}$$

where γ is a positive constant. One may observe that if we take the parameter $\gamma = 1$, the position-dependent mass is reduced to constant mass.

Using the identities $y = f(x)$ and $f'(x) = m^{\frac{1}{2}}(x)$, we obtain

$$y = f(x) = \int m^{\frac{1}{2}}(x)\,\mathrm{d}x = x + (\gamma - 1)\arctan x,\tag{22}$$

so that g becomes (vide equation (15))

$$g(x) = \sqrt{\frac{(1 + x^2)}{(\gamma + x^2)}}.\tag{23}$$

Here one can also see that $\gamma = 1$ then $y = x$.

Substituting the expression $m(x)$ and its derivatives into (18), we get

$$\tilde{V}(x) = \frac{1}{2}\left(f(x)^2 + \frac{8(2f(x)^2 - 1)}{(2f(x)^2 + 1)^2}\right) + \frac{(\gamma - 1)(3x^4 + 2(2 - \gamma)x^2 - \gamma)}{2(\gamma + x^2)^4},\tag{24}$$

where $f(x)$ is as given in (22). Again in the choice $\gamma = 1$ the target potential $\tilde{V}(x)$ is reduced to the reference potential $V(y)$.

From the relations (6), (11) and (19), we can fix the eigenvalues and eigenfunctions for the potential (24) of the form

$$\tilde{\psi}_n(x) = N_n\sqrt{\frac{(\gamma + x^2)}{(1 + x^2)}}\,\frac{\mathcal{P}_n[f(x)]}{(1 + 2f(x)^2)}\,\mathrm{e}^{\frac{-f(x)^2}{2}},\tag{25}$$

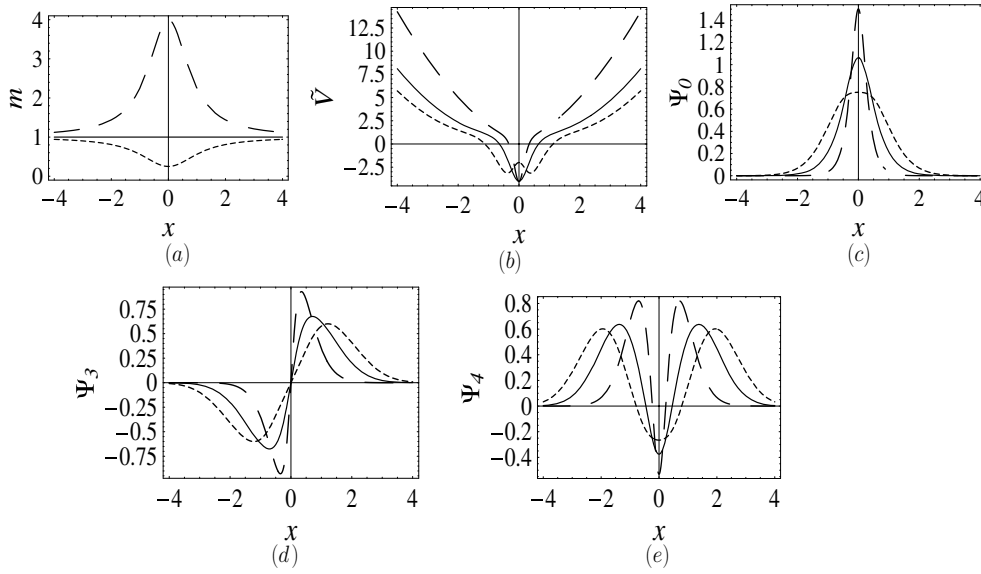$$\tilde{E}_n = -\frac{3}{2} + n,\qquad n = 0, 3, 4, \ldots,\tag{26}$$

Figure 1. The mass (*a*), target potential (*b*), wavefunction ψ_0 (*c*), wavefunction ψ_3 (*d*) and wavefunction ψ_4 (*e*) of Case 1. The solid lines represent $\gamma = 1$, the short dashed lines and long dashed lines represent the cases $\gamma = \frac{1}{2}$ and 2, respectively.

where the function $f(x)$ and normalization constant N_n are as given in (22) and (7), respectively. In figure 1, we plot the mass distribution $m(x)$ (vide equation (21)), target potential $\tilde{V}(x)$ (vide equation (24)) and the wavefunctions $\psi_n, n = 0, 3, 4$ (vide equation (25)) for three different values of the parameter γ, namely $\gamma = 0.5, 1$ and 2. The case $\gamma = 1$ corresponds to the constant mass case and is plotted in the form of solid lines in figure 1. The short dashed and long dashed lines represent the cases $\gamma = 0.5$ and 2 respectively for the mass distribution (21). The target potential $\tilde{V}(x)$ is symmetric and confining, and the shape of $\tilde{V}(x)$ becomes sheer as γ increases while keeping the similarity in its shape to that of the reference potential $V(x)$. The wavefunctions also keep their shape as that of the source potential.

4.2. Case 2

Now we consider a second mass profile which is found to be useful for studying transport properties in semiconductors [6], namely

$$m(x) = 1 + \tanh(\gamma x), \tag{27}$$

which becomes abrupt as γ becomes large. For $\gamma = 0$ the position-dependent mass m is reduced to the constant mass case. This mass example was also considered by Dekar *et al* [33] for a potential that has the same shape of a smooth step. Here we are interested in bound states.

As we have done in the previous case, first let us fix the mapping function y by using the identity $y = f(x) = \int m^{\frac{1}{2}}(x) \, dx$ which turns out to be

$$y = f(x) = \left(\frac{\sqrt{2}}{\gamma}\right) \operatorname{arctanh}\left(\frac{\sqrt{1 + \tanh(\gamma x)}}{\sqrt{2}}\right). \tag{28}$$
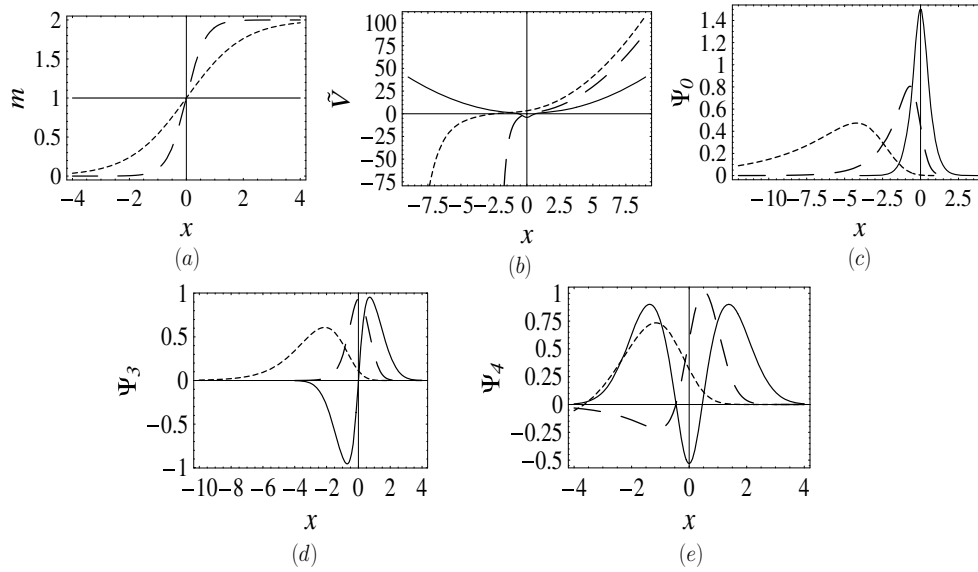
Figure 2. The mass (*a*), target potential (*b*), wavefunction ψ_0 (*c*), wavefunction ψ_3 (*d*) and wavefunction ψ_4 (*e*) of Case 2. The solid lines represent $\gamma = 0$, the short dashed lines and long dashed lines represent the cases $\gamma = \frac{1}{2}$ and 1.5, respectively.

In the present case, we have $f(-\infty) = 0$ and $f(\infty) = \infty$ which in turn modifies the normalization constant as we pointed earlier. The new solvable potential $\tilde{V}(x)$ can be derived by substituting the mass distribution (27) into (18). Doing so, we obtain

$$\tilde{V}(x) = \frac{1}{2}\left(f(x)^2 + \frac{8(2f(x)^2 - 1)}{(2f(x)^2 + 1)^2}\right) + \frac{\gamma^2(-7 + 6\tanh(\gamma x) + \tanh^2(\gamma x))}{32(1 + \tanh(\gamma x))}. \tag{29}$$

Here one can also note that when $\gamma = 0$, the target potential is reduced to the reference potential. The eigenfunction turned out to be

$$\tilde{\psi}_n(x) = \tilde{N}_n(1 + \tanh(\gamma x))^{\frac{1}{4}} \frac{\mathcal{P}_n[f(x)]}{(1 + 2f(x)^2)} e^{\frac{-f(x)^2}{2}}, \tag{30}$$

$$E_n = -\frac{3}{2} + n, \qquad \tilde{N}_n = \left(\frac{(n-1)(n-2)}{2^{n-1}n!\sqrt{\pi}}\right)^{\frac{1}{2}}, \qquad n = 0, 3, 4, \ldots, \tag{31}$$

where $f(x)$ is as given in (28). In figure 2, we plot mass distribution (vide equation (27)), target potential (vide equation (29)) and the wavefunctions (vide equation (30)) for three different values of the parameter γ, namely $\gamma = 0, 0.5$ and 1.5. Unlike the earlier cases, for $\gamma > 0$, the target potential is neither symmetric nor antisymmetric. The wavefunctions have no definite parity. Again the solid lines represent the constant mass case, and the short dashed and long dashed lines represent the cases $\gamma = 0.5$ and 1.5 respectively. Finally, we note here that the wavefunctions ψ_i, $i = 0, 3, 4$, given in solid lines differ in size from those given in the other two cases, namely Cases 1 and 3. This is because of the fact that the normalization constant in the present case is $\sqrt{2}$ times than that of the other two cases.
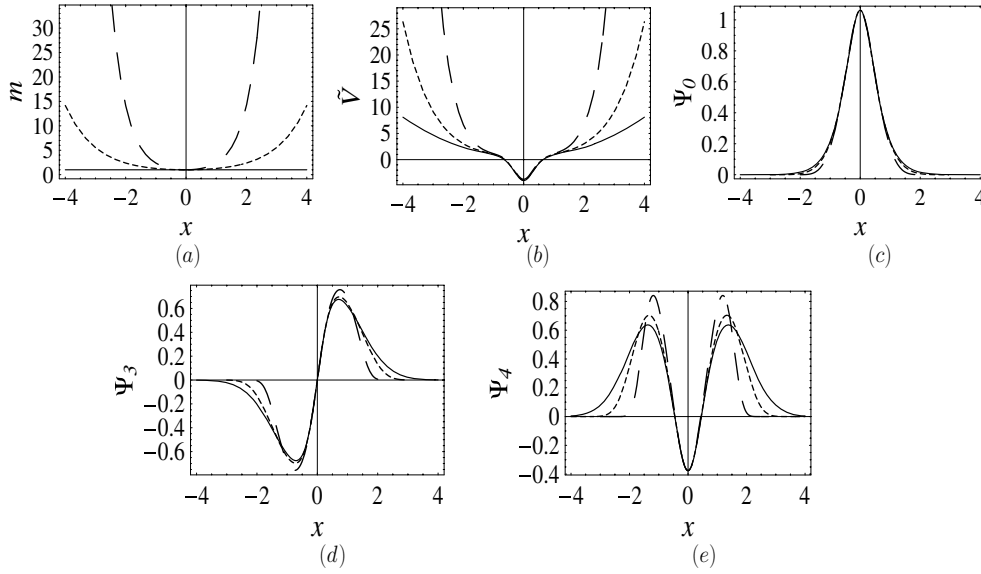
Figure 3. The mass (*a*), target potential (*b*), wavefunction ψ_0 (*c*), wavefunction ψ_3 (*d*) and wavefunction ψ_4 (*e*) of Case 3. The solid lines represent $\gamma = 0$, the short dashed lines and long dashed lines represent the cases $\gamma = \frac{1}{2}$ and 1, respectively.

4.3. Case 3

Another important mass profile which has been studied in graded alloys is of the form [34]

$$m(x) = \cosh^2(\gamma x) \tag{32}$$

so that for $\gamma = 0$ one can recover the constant mass case. A graded alloy quantum well, typically based on $Al_x Ga_{1-x} As$, will provide equispaced levels if the grading function, that is the variation of the mole fraction x along some direction, is chosen to be parabolic [34]. Such a design of quantum well structures with some number of equispaced levels enables resonant interaction at all levels with monochromatic light and fully resonant interaction. We intend to construct the wavefunctions and energy values for the PDMS with this mass profile.

Repeating the procedure outlined in the previous two cases, we found that the mapping function turns out to be

$$y = f(x) = \int m^{\frac{1}{2}}(x)\,dx = \frac{\sinh(\gamma x)}{\gamma}. \tag{33}$$

In the present example, we have $f_{\min} = -\infty$ and $f_{\max} = \infty$.

The associated solvable potential takes the form

$$\tilde{V}(x) = \frac{1}{2}\left(f(x)^2 + \frac{8(2f(x)^2 - 1)}{(2f(x)^2 + 1)^2}\right) - \left(\frac{\gamma^2}{16}\right)\operatorname{sech}^4(\gamma x)(3\cosh(2\gamma x) - 7). \tag{34}$$

The eigenfunction becomes

$$\tilde{\psi}_n(x) = N_n \cosh^{\frac{1}{2}}(\gamma x)\frac{\mathcal{P}_n[f(x)]}{(1 + 2f(x)^2)}\,\mathrm{e}^{\frac{-f(x)^2}{2}}, \tag{35}$$

$$E_n = -\frac{3}{2} + n, \qquad n = 0, 3, 4, \ldots, \tag{36}$$

where $f(x)$ and the normalization constant are as given in (33) and (7), respectively. In figure 3, we plot mass distribution (equation (32)), target potential (equation (33)) and the wave functions (equation (35)) for three values of the parameter, namely $\gamma = 0, \frac{1}{2}$ and 1. In this case, one can see that the target potential and the wavefunctions retain their shape as that of the original potential.

5. Conclusion

In this paper, we have considered the newly discovered Schrödinger solvable nonlinear oscillator that is related to the isotonic oscillator as the reference potential and obtained a class of exactly solvable potentials by transforming it into the PDMS. For our illustration, we have considered three different types of mass distributions which are often considered in semiconductor physics. We have also given bound state energies and corresponding wavefunctions for all three solvable potentials. As shown in this paper, the source potential and the target potential share the same bound state spectra.

Interestingly, one can also consider that the mass distribution is an exponential function of x, that is $m(x) = e^{\gamma x}$ or a quadratic power in x, that is $m(x) = (\alpha + \gamma x^2)$, where γ and α are constants. These kinds of mass functions had already been used in the study of confine energy states for carriers in semiconductor quantum well structures [6]. One may repeat the analysis given in this paper for the above mass distributions and identify new potentials that share the same energy spectra as that of the model (1).

Acknowledgments

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